geometry report of eighty pages was published in full in *School Science and Mathematics* in 1911, and again in final form in the *Mathematics Teacher* in 1912, and also in the proceedings of the National Education Association for 1911 and 1912. It was distributed through the Bureau of Education in Washington and has had a circulation, in this country and abroad, of 27,000 copies, and a request has just been granted to the Irish Journal of Education for its republication there in full.

The Underlying Causes. This decade of remarkable activity on the part of the teachers of mathematics in this country can be accounted for only on the hypothesis of a widespread and deep-seated feeling of unrest and dissatisfaction with the present methods and results, a conviction that in some way the mathematics teaching in the schools fails to connect vitally with the preparation for living in the twentieth century needed by the vast majority of the pupils in the schools. Possibly the following quotation from Professor Moore's address, above cited, may be the true analysis of the situation:

"He (Perry) asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies this logic; and moreover that he should be thoroughly *interested* in the subject studied. . . .

"As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry's just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods in general, in continuous relations with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with the subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much if not irreparable injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics in particular there is no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations, to be sure in a naïve and elementary way, of the finest mathematical tools which the centuries have forged, that under skillful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling toward his mathematics extremely different from that which is now met with only too frequently—a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions."

Whether or not this is the true interpretation of the present conditions, it is certainly significant that Professor Klein advocates practically the same remedy and that this is one of the questions which the International Commission has set for itself in its further deliberations.

The Indirect Results. Whatever may be the true causes of this widespread agitation and discussion, and whatever may be the final conclusions reached, this much is certain, that teachers of mathematics, the world over, are doing a vast amount of reading, investigating, experimenting, and careful thinking about the problems that confront them, and in this very attitude of mind lies the hope and the assurance that ultimately sane conclusions will be reached. In this country, especially, where there is a minimum of centralized authority, the final

disposition of these important questions depends largely upon such a development of thoughtful attention and investigation on the part of the general body of teachers as seems to be now in progress.

The indirect result, therefore, of all these discussions has been to stir teachers out of their lethargy and to encourage independent study outside of their routine duties. This seems to be the most plausible explanation of the vast increase in attendance upon the summer sessions at all the institutions where these are held. The report upon this topic[1] by the present writer, as a part of the American report of the International Commission, showed that the development of summer schools (that is, of summer sessions of normal schools, colleges and universities, conducted on the same basis as the work in any other part of the year), has been a remarkable phenomenon, and this has gone forward by leaps and bounds ever since that report was made. Along with the rapid increase of summer attendance in general, and of students of mathematics in particular, has gone the enrichment of the curricula, especially in the number and variety of courses which may count toward higher degrees, and in the courses on the history and teaching of mathematics. All this cannot fail to produce a profound impression upon those who are availing themselves of these opportunities, to arouse new enthusiasm, to stimulate fresh activity, to develop intelligent reaction on all the questions which confront the teachers of mathematics in this country.

The College Situation. For the most part, the foregoing data and observations apply to the teaching of mathematics in the secondary schools. All of the associations mentioned are primarily for secondary teachers, although many of the most enthusiastic members are college teachers. It is true that the investigations of the International Commission include schools of all grades from the elementary schools to the universities, but so far almost no attention has been given to the teaching of mathematics in the colleges, although it will readily be admitted that the problem is not so very different there, especially in first two years. It is practically certain that every charge that has been made against the teaching of mathematics in the secondary schools may be made with equal force in regard to the Freshman and Sophomore courses in college. In fact, there may be wider room for desirable modification in such subjects as analytic geometry and the calculus than in elementary algebra and geometry, if we may judge by some of the more recent texts and by the present practice of not a few colleges and universities.

But how shall the rank and file of teachers of college mathematics learn of the modifications in progress and of the success or failure of experiments in other institutions. Clearly the only way is by intercommunication, which must be either personal and hence very limited, or through some general medium of publication. It is precisely in this capacity that the editors of the MONTHLY believe its usefulness should be developed, and they are ready to devote any reasonable amount of space to the discussion of questions of this character.

[1] Published in the AMERICAN MATHEMATICAL MONTHLY, Vol. XIX, July, 1912. Afterwards also printed with the other reports in the Government bulletins.

[2] For one important exception see an item in the Notes and News of this issue.

However, it would not seem wise that such a journal should be devoted entirely or chiefly to pedagogical discussions as such. It may, and often does, happen that the stimulus derived from reading an article not directly pedagogical in its nature may produce a reaction strongly beneficial on the pedagogical side. Such, for instance, may be the effect of articles like those of Professors Hedrick, Huntington and Coolidge in the present issue, and such without doubt has been the effect of Professor Cajori's "History of Logarithms" which has been running since January. Even those articles which appear too abstruse for a given reader may provide just the stimulus which he particularly needs, not only to insure his continued mental activity but to actually furnish a form of mental stimulus which may be essential to his successful work as a teacher. This is the reason for publishing in the MONTHLY such articles as those of Professor Lehmer in the May issue, of Dr. Miles in the April issue, and of Professor Dickson in the March issue. (See the reference to this question under Notes and News.)

Conclusion. While it is true that agitation does not necessarily mean progress, it is also true that there is seldom any progress without agitation. We confidently believe that the unprecedented activity among teachers of mathematics during the past decade has resulted, and will further result, in substantial progress. It should be a keen incentive to every teacher that he or she may have some active part in our nation-wide determination to reconsider the foundations of our teaching and to improve our methods wherever possible; and that in this effort we are allying ourselves with a world-wide movement toward the same end.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

BY Fflorian CAJORI, Colorado College.

V. GENERALIZATIONS AND REFINEMENTS EFFECTED DURING THE NINETEENTH CENTURY.

THE GENERAL POWER AND LOGARITHM.

We have seen that the general theory of a^b, where both a and b are complex numbers, was outlined by L. Euler in his *Recherches sur les racines imaginaries des équations* of 1749, and that this paper failed to command the attention of mathematicians. We shall see now that three quarters of a century later the theory of the general power was elaborated by mathematicians of Germany, England, France and the Netherlands. At the opening of the nineteenth century this subject appeared difficult to many, as may be inferred from a paper of A. Q. Buée, in which the author contends that $\sqrt{-1}$ signifies perpendicularity,[1] and is finally led to the conclusion that $(\sqrt{-1})^n = \pm n(\sqrt{-1})$. The difficulty of the subject appears also in a paper of Argand who in 1813 ventured the statement that the expression $(\sqrt{-1})^{\sqrt{-1}}$ "would offer the simplest example of a

[1] *Philosoph. Trans.* for the year 1806, London, 1806, p. 67.

quantity irreducible to" the form $a + ib$.[1] Exception was taken to this statement by J. F. Français, professor in Metz, who pointed out that Euler had found $(\sqrt{-1})^{\sqrt{-1}} = e^{-1/2\pi}$, and showed that $(c\sqrt{-1})^{d\sqrt{-2}}$ is reducible[2] to $a + ib$. In the course of the next fifteen or twenty years sufficient familiarity was acquired with imaginaries to enable several mathematicians to grapple successfully with the theory of the general power. In the early history of the logarithms of positive numbers it was found surprising that logarithms were invented independently of exponents. Now another surprise is in store, namely that the theory of the general power is made to depend upon the theory of logarithms. All interpretations of a^b, where a and b are complex numbers, involve previously established results on logarithms. It thus appears that, historically, the logarithmic concept is the more primitive.

From the subtelty of general logarithmic theory arises the danger of its occasional recrudescence. We give an instance of this before we proceed to the general theory. Traces of confusion are found in an article by A. J. H. Vincent, professor at the Royal College at Reims—an article which possesses many points of merit. It touches upon the multiple nature of the logarithmic curve and its association with logarithmic theory. It is entitled, *Considérations nouvelles sur la nature des courbes logarithmiques et exponentielles.*[3] The author lets the variable x in $y = a^x$ take fractional values with even denominators, and obtains the usual two real branches. Vincent's article is discussed by George Salmon in his *Higher Plane Curves* (1879, page 286). Salmon accepts Vincent's idea that the curve has multiple branches. Vincent defines $1/n$ as the logarithm of all the numbers $a^{1/n}$, and concludes that, if a is positive, some $-$ numbers have real logarithms; if a is negative, some $+$ and some $-$ numbers have no logarithms, and so on. Vincent studies the discontinuities of his curves, but does not inquire into the consistency of his multiple-definition of logarithms. Peacock expresses himself as follows:

> "The question of the identity of the logarithms of the same number, whether positive or negative, . . . has been frequently resumed in later times. The arguments in favour of the affirmative of this proposition, which were for the most part founded upon the analytical interpretation of the properties of the hyperbola and logarithmic curve, were not entitled to much consideration, in as much as they were not drawn from an analysis of the course followed in the derivation of the symbolical expressions themselves and from the principles of interpretation which those laws of derivation authorized."[4]

The Vincent article was criticized by J. P. W. Stein,[5] who took the view that every fractional index may be converted into one having an even denominator so that there would be a double number corresponding to *every* logarithm. Vincent was criticized also by D. F. Gregory who said:[6]

> "It might, perhaps, have weakened his belief in the correctness of the results, if he had come to the conclusion, as he ought to have done, that the same logarithm corresponded to positive, negative, and impossible quantities. These last he seems quite to have overlooked."

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[1] Gergonne, *Annales de Math.*, T. IV., 133–147.
[2] Same journal, Vol. IV, pp. 71–73.
[3] *Gergonne's Annales de math.*, Nismes, 1824 et 1825, T. XV., pp. 1–38.
[4] *Report of British Ass'n*, London, 1834, p. 266.
[5] Gergonne, *Annales de math.* T. XV, p. 231.
[6] *Cambridge Math. Jour.*, Vol. I, 1837, "On the Impossible Logarithms of Quantities."

A more serious recrudescence is seen in an article by L. C. Bouvier,[1] ex-officier de génie. In 1823 he derived the eulerian formulæ for $\log x$ and $\log(-x)$ from the relation $\log x = n(\sqrt[n]{x} - 1)$, $n = \infty$. Then, solving for x, he got $x = \left(1 + \frac{1}{n}\log x\right)^n$, *d'où, à cause de n infini, on peut conclure* $x = \left(1 + \frac{1}{n}\log x\right)^{n+1/k}$ $= \left(1 + \frac{1}{n}\log x\right)^n \cdot \left(1 + \frac{1}{n}\log x\right)^{1/k} = N\sqrt[k]{1}$, N being real and k any positive integer. Thus, not only is the logarithm of a number many-valued, but for any one logarithm the anti-logarithm is many-valued. For $k = 2$, $\log x = \log(-x)$. *Ces considérations nous semblent de nature à terminer, une fois pour toutes, le différend qui s'est élevé autrefois entre Euler et d'Alembert, sur la nature des logarithmes des quantités négatives.* Bouvier made no attempt to test the inherent consistency of his system. That there was inconsistency was claimed by J. P. W. Stein,[2] professor at the gymnasium at Trèves. Stein pointed out that Bouvier could not accept the eulerian formulæ, $\log x = r + 2m\pi\sqrt{-1}$, ($x$ positive) and $\log x = r + (2m+1)\pi\sqrt{-1}$ (x negative), and at the same time take $x = N\sqrt[k]{1}$. Starting into an untrodden path, Stein advances new arguments to show that Euler's formulæ are incorrect. From the relation $e^{r+z\sqrt{-1}} = y$, r being the real logarithm of y, z a number to be determined, Stein first deduces $e^{z\sqrt{-1}} = 1$ and from this obtains Euler's formulæ. *Mais il faut observer que, si* e^r *peut admettre plusieurs valeurs différentes*, say k values, then $e^r = y[\cos(2p\pi/k) + \sqrt{-1}\sin(2p\pi/k)]$. This result, combined with the eulerian values, yielded him $\log y = r + [2m - (2p/k)]\pi\sqrt{-1}$, $\log(-y) = r + [2m' + 1 - (2p'/k)]\pi\sqrt{-1}$. When k is even, he obtained $\log y = \log(-y)$. We see that, in Stein's system, the relation $x = \log y$ is no longer defined by the simple equation $e^x = y$, but by the equation $e^{x-(2p\pi/k)} = y$. The general expression for the logarithm of a number involves two independent arbitrary constants, m and p, instead of one only, as in the eulerian system. More perplexing is the circumstance that the number of values k which e^r takes, was seemingly not supposed to be the same in all cases or for all numbers. The author does not explain how, under these conditions, his system can yield a theory of logarithms that is general, consistent and useful. Certainly Stein did not move in the direction of Cauchy, who, as we shall see, by the introduction of principal values, aimed at greater simplicity and the inauguration of law and order.

Proceeding to the general theory of a^b, we observe that an author of very marked influence in the development of this theory was Martin Ohm, professor in Berlin. In 1811, when he was privat-docent at the university of Erlangen, he conceived the idea of writing the system of mathematics which appeared later under the title *Versuch eines vollkommen consequenten Systems der Mathematik*, Nürnberg, 1822–1852. The aim of the work is indicated by its title. It has been much criticized, but the part on the general power and logarithms is meri-

[1] Gergonne, *Annales de math. p. et appl.*, T. XIV, 1823–4, p. 275.
[2] Same journal, T. xv, p. 110.

torious. These topics are treated in the second volume, of which the second edition, Berlin, 1829, lies before us. The first edition appeared in 1823. The edition of 1829 is the one usually quoted by contemporary writers. Ohm gave an exposition of his theory of logarithms as early as 1821, in a Latin thesis which had probably a very limited circulation.[1] We have never seen it quoted anywhere. Ohm introduces the terms *complete equation* and *incomplete equation*. An equation is *complete* when both sides of it have the same number of values representing exactly the same expressions; it is *incomplete* when one or several (but not all) of the many-valued expressions on the right and left are the same.[2] We have pointed out that Euler had occasion to consider these two types of equations in his paper of 1749 on logarithms. After having developed the eulerian theory of logarithms Ohm takes up the general power,[3] "allgemeine Potenz," a^z, where both a and x are complex numbers, namely $a = p + qi$, $x = \alpha + \beta i$. Assuming e^z as always single-valued, and letting $r = \sqrt{p^2 + q^2}$, $\log a = Lr + (\pm 2m\pi + \varphi)i$, he takes $a^x = e^{x \log a} = e^{\alpha \cdot Lr - \beta(\pm 2m\pi + \phi)} \cdot \{\cos[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)] + i \sin[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)]\}$, where $m = 0, +1, +2, \cdots$ and L signifies the tabular logarithm. Thus the general power has an infinite number of values, but all are of the form $a + bi$. Ohm shows (1) that all of the infinite values are equal when x is an integer, (2) that there are n distinct values when x is a real, rational fraction s/n, (3) that some of the values are equal, though the number of distinct values is infinite, when x is real but irrational, (4) that the values are all distinct when x is imaginary. His idea of the irrational is partly explained by the statement that, if $x = s/n$ is irrational, then s and n are *unendlich grosse und nie angebbare Zahlen*. As illustrations of what his formula yields in special cases, Ohm shows that $i^i = (-i)^{-i} = e^{-(\pm 2d + \frac{1}{2})\pi}$, $d = 0, 1, \cdots$.

He inquires next, how the formulæ (*A*) $a^x \cdot a^y = a^{x+y}$, (*B*) $a^x \div a^y = a^{x-y}$ (*C*) $a^x \cdot b^x = (ab)^x$, (*D*) $a^x \div b^x = (a \div b)^x$, (*E*) $(a^x)^y = a^{xy}$ apply to the general exponent a^z, and finds that (*A*), (*B*) and (*E*) are incomplete equations, since the left members have "many, many more" values than the right members, although the right-hand values (infinite in number) are all found among the "infinite times infinite" values on the left; that (*C*) and (*D*) are complete equations for the general case.

Ohm next sets limitations upon the choice of values of *log a* and *log b*. When out of the infinite number of values of *log a*, some particular one, say α, is to be selected in the power $a^x = e^{x \log a}$, he indicates this by the notation $(a||\alpha)^x$. He writes similarly, $(b||\beta)^x$. With this understanding, we can write $x \log a + y \log a = (x + y) \log a$, and formula (*A*) becomes complete. The same is true of (*B*); while (*C*) and (*D*) remain complete equations under this restricted interpretation. It will be noticed that Ohm did not introduce the particular

[1] *De innumerosis novis logarithmorum generibus. Dissertatio qua ad examen solemne in gymnasio regio Thorunensi die XIV. Aprilis MDCCCXXI publice habendum omnes literarum cultores invitat Dr. Martinus Ohm, nonnullarum societatum literarum sodalis.* Berolini, Typis Haynianis.

[2] M. Ohm, *System d. Math.* 2, Theil, Berlin, 1829, p. 386.

[3] Ohm, *op. cit.*, 2. Theil, 1829, p. 412.

value of a^z, which is now called the "principal value." Aside from that his treatment of the general power is mainly that of the present time, except, of course, in the explanation of the irrational.

Ohm proceeds to the general logarithm, and he makes the novel and significant statement that "with the concept of the general power is given the concept of the general logarithm $b?a$, if by this is meant every expression x, such that one has $a^x = b$ or $e^{x \log a} = b$."[1] Notice his notation for the general logarithm, $b?a$. As x has an infinite number of values for each value of $\log a$, we have here the appearance of two independent and arbitrary constants, as in the researches of Stein and in those of Graves, Hamilton, and others, to be discussed later. He says that since b and a are taken completely general, and a^x has an infinity of values, $b?a$ is wholly undetermined, unless we state for what value of *log* a the power a^x is to be taken. He adopts the special notation $b?(a \,||\, \alpha)$, which means the logarithm of b to the base a, when $\log a = \alpha$. Ohm shows that $b?(a \,||\, \alpha) = (\log b)/\alpha$ is a complete equation. If a is positive, say e, then Ohm's logarithmic system reduces to the familiar eulerian form. "And if a is not positive, then the general logarithm, as well as each of its special cases is different from every one of the logarithms hitherto defined and discussed."

It must be granted that Ohm has surpassed all his predecessors in the generality and fullness of discussion of the expression a^z, and that he is the first writer to successfully base the general theory of logarithms (having a complex number as a base) fully and unreservedly upon the theory of the general power a^z. But the theory of the general power, as we have seen, is not developed independently of logarithms; in fact it uses the theory of logarithms of complex numbers to the base e. Thus, the eulerian logarithms have answered as a step-ladder leading to the theory of the general power; the theory of the general power, in turn, has led up to a more general theory of logarithms having a complex base.

The first original research on the logarithms of complex numbers published in England appeared in 1829, in the *Philosophical Transactions*, from the pen of John Graves, then a young man of 23. Graves was a class-fellow of William Rowan Hamilton in Dublin. Hamilton repeatedly expresses his indebtedness to Graves and states that reflecting on Graves's ideas on imaginaries led, finally, to his invention of quaternions. Graves became a noted jurist. His paper of 1829 bears the title: "An attempt to rectify the inaccuracy of some logarithmic formulæ." The author modifies $l1$ by certain rather startling extensions, yielding $l1 = (2m'\pi\sqrt{-1})/(1 + 2m\pi\sqrt{-1})$. This is done by taking the base e not in its *arithmetical* form, but in its more general form $e^{1+2m\pi\sqrt{-1}}$, so that x is defined as the general logarithm of y, not by the equation $e^x = y$, but by the equation $e^{(1+2m\pi\sqrt{-1})x} = y = e^{\log y + 2m'\pi\sqrt{-1}}$, where $\log y$ means the real or tabular logarithm of y. Letting $y = 1$ gives $l1$ as written above. Thus, Graves (like Ohms) claimed that in the general expression for the logarithm there are two arbitrary and independent integers, m and m', instead of simply one, as given by Euler.

[1] Ohm, *op. cit.*, 2. Theil, Berlin, 1829, p. 438.

In a paper of subsequent date, published in the same volume of the *Philosophical Transactions* (1829), the Rev. John Warren of Cambridge contributed a paper which we noticed when we were considering the geometric representation of algebraic quantities. Warren arrived at some of the algebraic results of Graves. In June, 1832, Vincent published, at Lille, results identical in effect with the principal formulæ of Graves.[1]

For lack of explicitness in Graves's writings exception was taken to his views by Augustus De Morgan in his treatise on the Calculus of Functions, sections 158 and 245, published in the *Encyclopaedia Metropolitana.*

In 1833 George Peacock made a report on the recent progress of certain branches of analysis in which he touches upon eulerian logarithms,[2] and argues that even in that theory negative numbers may have real logarithms. For, consider $-a^m$ as originating from $(-1)(+a)^m$, then $l(-a)^m = (2r+2mr'+1)\pi\sqrt{-1} + m \log a$. If we suppose $m = \frac{1}{2}$, $r = 0$, $r' = -1$, we have $l(-\sqrt{a}) = \frac{1}{2}\log a = l\sqrt{a}$. The reader will see that the eulerian theory is here violated by the assumption that in $2mr'$ it is possible to take r' odd. As regards the work of Graves, Peacock thought that there was a fundamental error in his generalization, because he makes a *periodic* quantity the base of his system. Graves sent a defence to the British Association in 1834, also to the *Philosophical Magazine.*[3] The outcome of the discussion was that Graves withdrew the statement contained in the title of his first paper, to the effect that he was attempting to "rectify the inaccuracy" of the eulerian theory, while De Morgan admitted that if Graves desired to extend the idea of a logarithm so as to include not only the logarithms of the arithmetical form of the base, but also those of a more general form, there was no error involved in the process. In fact, De Morgan suggested a still further extension.[4] Logarithmic systems like that of Graves, in which there are two arbitrary and independent integers were also worked out by Sir William R. Hamilton.[5]

Without being aware, apparently, that De Morgan and Graves had come to an understanding, D. F. Gregory of Trinity College, Cambridge, published an able article "On the Impossible Logarithms of Quantities,"[6] in which he inquires "which is the correct result," that of Graves or of his opponents. Though admitting the possibility of both, Gregory considers the one with an arithmetical base as more expedient. Gregory also points out that Vincent, Peacock and Graves, each erroneously thought he had established that in certain cases there is a logarithm in common for positive and negative numbers. Gregory died at the premature age of 31, yet in his short scientific career he did much toward establishing the foundations of algebra.

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[1] *Report of the Fourth Meeting of the Brit. Assn.*, London, 1835, p. 524. We have not seen Vincent's publication of 1832.

[2] *Report of the Third Meeting of the British Ass'n*, London, 1834, p. 264.

[3] *Phil. Mag.*, Vol. VIII, 1836, p. 281.

[4] *Philos. Magazine*, Vol. IX, 1836, p. 252.

[5] "On Conjugate Functions or Algebraic Couples, etc.," *Transactions of the Royal Irish Acad.*, Vol. 17, Part II, 1835; *Lectures on Quaternions*, Dublin, 1853, preface, p. (12).

[6] *Cambridge Math'l Jour.*, Vol. I, 1837, p. 226.

Deep philosophic insight and logical power are evident in four papers *On the Foundation of Algebra*, published by Augustus De Morgan in 1842 and 1849. They were read before the Cambridge Philosophical Society in 1839, 1841, 1843 and 1844, respectively. From his first paper we quote the following:[1]

"If we define $\log x$, or rather λx (reserving $\log x$ for the numerical logarithm of the length) to be any legitimate solution of $e^{\lambda x} = x$, it is plain that the logarithm of n inclined at an angle ν, (or of N) to the base b inclined at an angle β, (or B) is to be derived (avoiding ambiguity) from

$$(be^{\beta\sqrt{-1}})^x = = ne^{\nu\sqrt{-1}},$$

or

$$\lambda_B N = = \frac{\log n + \nu\sqrt{-1}}{\log b + \beta\sqrt{-1}}.$$

This result is real when $\log n/\log b = \nu/\beta$; nor is it more surprising that an impossible quantity (hitherto so called) should have a possible logarithm, than that exponential operations not containing $\sqrt{-1}$, or not interchanging exponents of length and direction, should in certain cases enable us to pass from one line to another."[2]

It is worthy of note that in discussing the exponent $\sqrt{-1}$ on page 184, the author discloses, though somewhat vaguely, the notion of abstract groups and develops the cyclic group of order four. This paper was written by De Morgan in 1839, or some years before Cayley and William Rowan Hamilton came out with their researches on groups. It is one of the earliest, perhaps the earliest, of English references to group theory.

De Morgan shows that Graves's result is a special case of the above. In his second paper, De Morgan proceeds to the study of a^b. Reasoning geometrically as before and letting (r, ρ) be a line of r units inclined to the unit line at an angle ρ, he calls the *logometer* of (r, ρ) or $\lambda(r, \rho)$, the logarithm of that line, not only with respect to its length, but also the quantity of revolution by which it attained its present direction. From $e^{\pi\sqrt{-1}} = -1$ he gets $\pi = \lambda(-1)/(\sqrt{-1})$,

"a proposition which, not many years since, was one of the mysteries of analysis. It is now a very simple geometrical proposition: the first side means a line of π units laid down positively on the unit-line; the second side means the logometer of a negative unit turned back through a right angle. Now the logometer of a negative unit is a line of π units erected positively perpendicular to the unit-line: whence the identity of the two sides is manifest."[3]

These subjects are elaborated also in De Morgan's *Double Algebra*, London, 1849. He aimed to avoid the consideration of ambiguous values of symbols, "a thing for which there is no necessity."

"The more I think on this subject, the better satisfied do I feel, that the new algebra should have no symbols of double or multiple value whatsoever; that is, that the meaning of each elementary symbol should not be considered as complete, unless it expresses the amount of revolution from the unit line by which it is to be made to attain its direction, as well as that direction itself."[4]

He points out that the consequences of the full extension of a^b turn out to be capable of expression by the particular case in general use. "Whereas,[5] in the common system ϵ and $\epsilon^{\sqrt{-1}}$ are the logarithmic bases employed for ordinary and periodic magnitudes, we have, in the system above described, employed $e^{(m+n\sqrt{-1})^{-1}}$, and $e^{(\nu-\mu\sqrt{-1})^{-1}}$."

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[1] *Trans. of the Cambridge Philos. Society*, Vol. 7, 1842, p. 186.

[2] The double sign of equality is used by De Morgan to indicate "that every symbol shall express not merely the length and direction of a line, but also, the quantity of revolution by which a line, setting out from the unit line, is supposed to attain that direction."

[3] *Trans. Cambr. Phil. Soc.*, Vol. 7, 1842, p. 294.

[4] De Morgan, *Trans. Cambr. Phil. Soc.*, Vol. 7, p. 296.

[5] *Trans. Cambr. Phil. Soc.*, Vol. VIII, Pt. II, p. 3.

An independent discussion of the generalized power $A^B = C$ is given by G. M. Pagani,[1] professor at the Catholic University of Löwen. He takes $A = \mu e^{\theta i}$, where the argument θ is called the *déterminatif* of A. The consideration of the *déterminatif* of a quantity is the only means, says the author, of avoiding paradoxes in algebraic analysis. He uses the notation $(a)^{m/n}$ to indicate all the n values of $a^{m/n}$. Assuming A and C as given, he calculates B. Letting $A = \mu e^{i\theta} = e^{l\mu + i\theta}$, $B = \alpha + i\beta$, he gets $C = e^{(l\mu + i\theta)(\alpha + i\beta)} = re^{i\phi}$ and finally $\alpha = (l\mu + \theta\phi) \div (l^2\mu + \theta^2)$, $\beta = (\varphi l\mu - \theta l\gamma) \div (l^2\mu + \theta^2)$. Here l means the tabular logarithm to the base e; θ and ϕ are each infinitely many-valued. Pagani points out that, if A is real and positive, so that $\theta = 2k\pi$, and if $\varphi = 2k\pi \cdot lr \div l\mu$, then the imaginary number $re^{i\phi}$ has a real logarithm. This conclusion is evidently erroneous. He says that his formulæ are *plus générales et par conséquent plus exactes* than those of Euler, since Euler's are obtained by letting $k = 0$. Pagani's paper was read before the Academy at Bruxelles on October 7, 1837. After the lecture Quetelet pointed out that J. S. Cerquero,[1] director of the observatory at San-Fernando, near Cadix, had reached similar results.[2] Later, Pagani prepared a note[3] on the limit of $(1+(x/n)^n$, x being complex, whereupon he was informed of the work of Graves in England on A^B. We shall see that the general power was treated also by Cauchy in 1847.[4]

Pagani's error, referred to above, arose from the tacit assumption that $(A^x)^y = A^{xy}$ is a complete equation. Ohm had shown that it is incomplete. A failure to pay attention to this matter leads to curious paradoxes. Th. Clausen of Altona gave such a paradox in 1827.[5] It was quoted by Peacock in his report.[6] When n is an integer, $e^{2n\pi i} = 1$, $e^{1+2n\pi i} = e$, hence also $e^{(1+2n\pi i)^2} = e = e^{1+4n\pi i - 4n^2\pi^2}$. Since $e^{1+4n\pi i} = e$, it would follow that $e^{-4n^2\pi^2} = 1$, which is absurd. In more condensed form the paradox is presented by E. Catalan[7] thus: $e^{2m\pi i} = e^{2n\pi i}$, where m and n are any two distinct integers. Raising both sides to the power $i/2$, we have the absurdity, $e^{-m\pi} = e^{-n\pi}$.

The reader will notice that $(e^{2m\pi i})^{i/2} = e^{-m\pi}$ is an incomplete equation; all values of the right member are values of the left member, but not *vice versa*. Let $((a))^x$ signify all the values of the general power. Since $((a))^x = e^{x \log a + 2p\pi i}$ is a *complete* equation, where $\log a = la + 2q\pi i$, la = tabular logarithm, $\pm p = 0, 1, 2, \cdots$ and $\pm q = 0, 1, 2, \cdots$, we obtain the following complete equations $(((e))^{2m\pi i}))^{i/2} = ((e^{2m\pi i \log e + 2p\pi i}))^{i/2} = e^{-m\pi - p\pi - 2mq\pi^2 i}$. When $p = q = 0$, we have the particular value $e^{-m\pi}$. Similar results are obtained for $(e^{2n\pi i})^{i/2}$. It follows that $(e^{2m\pi i})^{i/2} = (e^{2n\pi i})^{i/2}$ is an incomplete equation, in which the

[1] *N. mémoires de l'academie royale des sciences et belles-lettres de Bruxelles*, T. XI, Bruxelles, 1838, pp. 1–11.

[2] We have been unable to find any other reference to Cerquero's research.

[3] *Bulletins de l'académie r. d. scien. et b. l. de Bruxelles*, T. 6 (*1re partie*), *Bruxelles*, 1839, p. 256.

[4] Cauchy, *Exercices d'analyse et de phys. math.*, T. IV, 1847, p. 255.

[5] *Crelle's Journal*, Vol. 2, Berlin, 1827, pp. 286–287.

[6] *British Ass'n Report*, London, 1834, p. 347.

[7] *Nouv. Ann. de Math.*, 2e S., Vol. 8, 1869, p. 456. See also Vallès in the same journal, 2e S., Vol. 9, 1870, p. 20.

particular value e^{-mz} on the left is not equal to the particular value e^{-nz} on the right. This last equation becomes complete under the restrictions $m + p = n + p'$ and $mq = nq'$, where the integers p' and q' are used in the complete value of the right member in the same way as p and q were used on the left. De Morgan draws upon the notion of direction of lines to explain paradoxes of this character.[1] He reminds us that such fallacies have been seriously proposed as arguments against the use of imaginary quantities.

The generalizations of logarithms leading to systems having periodic or complex bases gave rise to formulæ of such complexity that the generalized system failed of adoption. The machinery was too intricate for general use. The theory of the general power a^b, a and b being complex numbers, was established satisfactorily with the aid of the eulerian logarithms to the base $e = 2.718$. No real advantage grew out of the new logarithmic systems. If in $a^z = c$, the complex numbers a and c are given and x is to be found, the eulerian logarithmic formulæ are sufficient for the solution. Thus it happened that the general logarithmic systems of Ohm, Graves, Vincent, Warren, De Morgan, Gregory, Hamilton and Pagani failed of recognition as useful mathematical inventions.

However, one result of the study of the general power and of logarithms having a number $a + ib$ as a base was of importance; this study led to the conclusion that no possible combination of real and ordinary complex numbers gives rise to new forms of numbers. De Morgan frankly admits that he had expected some "new *imaginary* or *impossible* quantities."[2] But the processes of addition and subtraction, multiplication and division, involution and its two inverses, evolution and logarithmation, constitute a closed group of operations; they complete the cycle of operations in the algebra of ordinary complex numbers. In the seventeenth century the logarithmation of negative and complex numbers was not recognized; hence the cycle of operations was not a closed one and the algebra of that time was not complete.

Similar researches were carried on for quaternions. We have seen that William Rowan Hamilton was in close touch with the work of J. T. Graves and De Morgan. He was familiar with the investigations of Ohm. Like Cauchy and others, Hamilton recognized the great inconvenience arising from the multiplicity of logarithmic values. ". . ., but it has been my object, in the present theory, to preclude, so far as I could, that indeterminateness by *definition*."[3]

A difficulty in Hamilton's development of general quaternion theory—his algebra of space—is described by him as follows:

> In the present theory of diplanar quaternions, we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and easier case of complanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value.[4]

More recently this difficulty has received the attention of Alexander Mac-

[1] De Morgan, *Double Algebra*, p. 136.
[2] *Trans. Cambr. Phil. Soc.*, Vol. VIII, Pt. II, 1844, pp. 3, 4.
[3] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, pp. (15), 556.
[4] W. R. Hamilton, *Elements of Quaternions*, London, 1866, Section 11, p. 386.

farlane. He points out that in the algebra of the plane there is no need of specifying the axis of rotation for a circular angle, such as e^{ib}. It is different in space. When angles on a sphere are considered, the axis must be inserted, and also $\pi/2$ for the indefinite i.[1] If by α^a we mean a radians round the axis α, α^1 means 1 radian round α. Here $\log \alpha^1 = \alpha^{\pi/2}$ and $\log \alpha^a = a \log \alpha^1 = a\alpha^{\pi/2}$. Whence $\alpha^a = e^{a \log_e a^1} = e^{aa^{\pi/2}}$. Now Hamilton adopted the principle which, in Macfarlane's notation is $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta + \sin\alpha\beta[\alpha\beta]^{\pi/2}$, where $[\alpha\beta]$ signifies the unit which is normal to α and β. Macfarlane, on the other hand, adopts the principle $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta - \sin\alpha\beta\,[\alpha\beta]^{\pi/2}$. By this modification Macfarlane is able to establish for his space-algebra the simple theorem which Hamilton could not derive from his assumption, $\log(e^{aa^{\pi/2}} \cdot e^{b\beta^{\pi/2}}) = a\alpha^{\pi/2} + b\beta^{\pi/2}$.

A SIMPLE FORMULA FOR THE ANGLE BETWEEN TWO PLANES.

By EDWARD V. HUNTINGTON, Harvard University.

Let us consider the familiar problem of finding the area of a surface by the methods of the integral calculus.

Let S be any closed portion of a surface whose equation is $z = f(x, y)$, and let ΔS be an " element " of the surface at any point P. Then, as is well known,

$$\Delta S = \frac{\Delta S'}{\cos\gamma},$$

where $\Delta S'$ is the projection of ΔS on the xy-plane, and γ is the angle between the xy-plane and the tangent plane to the surface at P. The area of S is therefore given by

$$S = \int\int \frac{dS'}{\cos\gamma},$$

where the integration is to be extended over the whole of the plane area S' which is the projection of S on the xy-plane.

The problem of finding the area S thus reduces to a simple problem in double integration, *provided the value of γ at every point of the surface is known.*

In the ordinary treatment of the subject in the text-books, the derivation of the formula for γ requires a considerable knowledge of analytical geometry of three dimensions, including the equation of a plane, the equations of a straight line normal to a plane, the method of finding the direction cosines of a line when its equations are given, the equation of a tangent plane, etc.

The object of this note is *to show how the angle γ may be found directly*, by very elementary methods. The formula obtained, in the easily-remembered form here given, and the methods of derivation are so simple that it is hardly possible that they are new; and yet I do not find them in any of the current text-books.

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[1] A. Macfarlane, "A vector-analysis as generalized algebra," *International Congress of Mathematicians*, Cambridge, August, 1912; "Account of researches in the algebra of physics," *Journal of the Washington Acad. of Sci.*, Vol. II, Nos. 14, 15, 16, 1912.

The Formula for the Angle γ.

Let P be any point of the surface, and let PA and PB (see either figure) be the lines in which the tangent plane at P intersects the vertical planes drawn through P parallel to the planes xz and yz respectively.

The angles φ_x and φ_y which PA and PB make with the horizontal can be readily determined from the fact that $\tan \varphi_x$ is the partial derivative of z with respect to x, and $\tan \varphi_y$ is the partial derivative of z with respect to y; thus,

$$\tan \varphi_x = f_x'(x, y) = D_x z, \quad \tan \varphi_y = f_y'(x, y) = D_y z.$$

We may therefore suppose that $\tan \varphi_x$ and $\tan \varphi_y$ are known for every point P of the surface.

Then if γ is the angle which the tangent plane at P makes with the base plane, the value of $\tan \gamma$ may be at once computed by the easily remembered formula

$$\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y,$$

whence, of course, $\sec \gamma = \sqrt{1 + \tan^2 \gamma} = \sqrt{1 + (D_x z)^2 + (D_y z)^2}$.

The proof of the formula may be obtained by either of two methods, as follows.

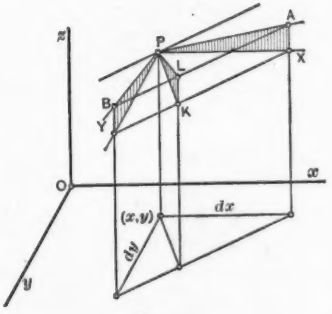


Fig. 1.

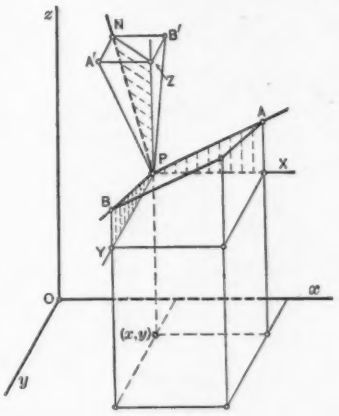


Fig. 2.

First Proof.—Referring to figure 1, we may suppose that the values of dx and dy have been so chosen that the perpendiculars AX and BY dropped from A and B on the horizontal plane through P shall be equal, say $AX = BY = h$. Then

$$\tan \varphi_x = \frac{h}{dx} \quad \text{and} \quad \tan \varphi_y = \frac{h}{dy}.$$

Now since $AX = BY$, the lines AB and XY will be parallel to the line of

intersection of the tangent plane with the horizontal plane. If, therefore, we draw PK perpendicular to XY and PL perpendicular to AB, then PK and PL will be perpendicular to the edge of the dihedral angle γ, so that the angle KPL will be the correct measure of that dihedral angle. Hence

$$\tan \gamma = \frac{h}{PK}.$$

But, by considering the right triangle PXY, we find at once that

$$\frac{1}{PK^2} = \frac{1}{dx^2} + \frac{1}{dy^2}.$$

Therefore

$$\tan^2 \gamma = \frac{h^2}{PK^2} = \frac{h^2}{dx^2} + \frac{h^2}{dy^2} = \tan^2 \varphi_x + \tan^2 \varphi_y,$$

which was to be proved.

Second Proof.—Through the given point P, in figure 2, draw PX, PY, PZ parallel to the coördinate axes, and let PN be the normal to the surface at P, so that $\angle ZPN = \gamma$.

In the plane PXZ draw PA' perpendicular to PA, so that $\angle ZPA' = \angle XPA = \varphi_x$, and in the plane PYZ draw PB' perpendicular to PB, so that $\angle ZPB' = \angle YPB = \varphi_y$; and let N, A', B', be the points of the lines PN, PA', PB' which lie in a horizontal plane through Z. Then

$$\tan \gamma = \frac{ZN}{PZ}, \qquad \tan \varphi_x = \frac{ZA'}{PZ}, \qquad \tan \varphi_y = \frac{ZB'}{PZ}.$$

If now we can show that the figure $ZA'NB'$ is a rectangle, we shall have

$$ZN^2 = ZA'^2 + ZB'^2,$$

and hence $\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y$, as desired.

We prove first that NA' is parallel to ZB'.

From the figure, PN is perpendicular to PA, PA' is perpendicular to PA, and PY is perpendicular to PA. Hence PA' lies in the plane PNY. But this plane cuts the horizontal plane through Z in a line NA' which is parallel to PY and hence parallel to ZB'.

Similarly we can prove that NB' is parallel to ZA'.

Hence the figure $ZA'NB'$ is at least a parallelogram. But the angle between ZA' and ZB' is known to be a right angle. Hence the figure is a rectangle, and the proof is complete.

It will be noticed that the distinction between the two methods of proof consists mainly in the fact that the required angle γ is regarded as the *dihedral angle* between the planes in the first proof, and as the *angle between the normals* to those planes in the second proof. These angles are of course equal.

A modification of the first proof may be obtained by considering *two* planes perpendicular to the edge of the dihedral, one containing AX and the other containing AY, instead of the single plane containing P.

A DIRECT DEFINITION OF LOGARITHMIC DERIVATIVE.*

By E. R. HEDRICK, University of Missouri.

1. The Usual Definitions. The logarithmic derivative of a function $y = f(x)$ is usually defined either as the derivative of the Napierian logarithm of y:

$$\frac{d \log_e y}{dx} = \frac{d \log_e f(x)}{dx}; \tag{1}$$

or by means of the resulting formula:

$$\frac{dy}{dx} \div y = f'(x) \div f(x). \tag{2}$$

These definitions suffer from the fact that the logarithm is defined only when $f(x)$ is positive, though (2) may be used independently of (1) when $f(x)$ is negative.

2. A Direct Definition. The importance of logarithmic differentiation in elementary work arises from its meaning as the *relative rate of increase* of y with respect to x. From this standpoint, the intervention of logarithms in the definition is wholly accidental and extraneous.

In order to define directly the relative rate of increase, we may divide the actual increase Δy by the increase in x, Δx, divide this ratio $\Delta y/\Delta x$ by some average value of $f(x)$ in the range Δx, and take the limit of this quotient as Δx approaches zero. Denoting the relative rate of increase by r_r, we may write

$$r_r = \lim_{\Delta x = 0} \left\{ \frac{\Delta y}{\Delta x} \div [\text{Average Value of } f(x)] \right\}. \tag{3}$$

The average value of $f(x)$ to be used may be selected in a variety of ways; that which seems most appropriate is the usual expression:

$$[\text{Average Value of } f(x)] = \frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x)dx,$$

where $x = a$ is the value of x at which the relative rate is to be found. Then (3) becomes

$$r_r = \lim_{\Delta x = 0} \left\{ \frac{\Delta y}{\Delta x} \div \left[\frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x)dx \right] \right\},$$

whence, simplifying and replacing Δy by $f(a + \Delta x) - f(a)$, we obtain

$$r_r = \lim_{\Delta x = 0} \left[\frac{f(a + \Delta x) - f(a)}{\int_a^{a+\Delta x} f(x)dx} \right]. \tag{4}$$

* Read before the Southwestern Section of the American Mathematical Society, November, 1912.

Let $\phi(x)$ be any indefinite integral of $f(x)$; then

$$\int f(x)dx = \phi(x), \qquad \int_a^{a+\Delta x} f(x)dx = \phi(a+\Delta x) - \phi(a),$$

and (4) becomes

$$r_r = \lim_{\Delta x=0} \left[\frac{f(a+\Delta x) - f(a)}{\phi(a+\Delta x) - \phi(a)} \right], \tag{5}$$

which we shall adopt as *the definition of the relative rate of increase of* $f(x)$, *or the logarithmic derivative of* $f(x)$, *with respect to* x.

3. Identification with the Usual Definition. At least when $f(x)$ is positive, the definition (5) coincides with the usual definition (1); and (5) coincides with the definition (2) whenever (2) has a meaning.

For, since

$$\frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\phi(a+\Delta x) - \phi(a)} \cdot \frac{\phi(a+\Delta x) - \phi(a)}{\Delta x} \tag{6}$$

and since the limit of $[\phi(a+\Delta x) - \phi(a)]/\Delta x$ surely exists and is equal to $f(a)$, if $f(x)$ is continuous at $x = a$, we have

$$f'(a) = r_r \cdot f(a) \tag{7}$$

provided $f'(a)$ exists.

Conversely, if r_r, as defined by (5), exists, and if $f(x)$ is continuous at $x = a$, it follows from (6) that $f'(a)$ exists.

4. The Law of the Mean. If $f(x)$ is continuous, and if r_r exists, the derivative $f'(x)$ exists; moreover $\phi'(x)$ exists and is equal to $f(x)$; hence, by the generalized law of the mean,*

$$\frac{f(a+\Delta x) - f(a)}{\phi(a+\Delta x) - \phi(a)} = \frac{f'(c)}{\phi'(c)} = \frac{f'(c)}{f(c)}, \tag{8}$$

where c lies between a and $a + \Delta x$.

The expression on the left, $[f(a+\Delta x) - f(a)] \div [\phi(a+\Delta x) - \phi(a)]$, may be called the average relative rate over the interval Δx; with this notation, the equation (8) may be stated as follows: *The average relative rate over the interval* Δx *is precisely equal to the instantaneous relative rate at some point in that interval.*

This statement, which is the analogon to the law of the mean for ordinary derivatives, may be used to prove directly several important theorems.

Thus, *if the logarithmic derivative is zero identically, it follows from* (8) *that* $f(x)$ *is a constant.*

Again, suppose that the logarithmic derivatives of two functions $f(x)$ and $F(x)$ are identically equal; then

$$0 = \frac{f'(x)}{f(x)} - \frac{F'(x)}{F(x)} = \frac{f'(x)F(x) - F'(x)f(x)}{[F(x)]^2} \div \frac{f(x)}{F(x)}.$$

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 8; it is vital that c is the same in numerator and denominator.

The extreme right-hand member is, however, the logarithmic derivative of $f(x)/F(x)$; hence by the preceding theorem, $f(x)/F(x)$ is constant, or $f(x) = k \cdot F(x)$: *If two functions have the same logarithmic derivative, their quotient is a constant*; or *If the logarithmic derivative of a function is given, that function is determined except for a constant factor.*

These are the fundamental theorems concerning logarithmic derivatives; they are precisely analogous to the corresponding theorems for ordinary derivatives.

5. Discontinuity. It has been shown above that $f'(x)$ exists whenever the relative rate r_r defined by (5) exists, provided only that $f(x)$ is continuous. It is, however, possible that $f'(x)$ and therefore also r_r is discontinuous.

It might appear that (8) precludes such a possibility, for the left side approaches r_r, and c approaches a as Δx approaches zero. But c does not necessarily take on all values; hence the approach of c to a may be only through a set of special values of x.

That the possibility just mentioned actually does occur is manifested by the example

$$f(x) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} + 1, \qquad x \neq 0; \qquad f(0) = 1;$$

which has an indefinite integral:

$$\phi(x) = x^4 \sin\frac{1}{x} + x, \qquad x \neq 0; \qquad \phi(0) = 0.$$

Its derivative is

$$f'(x) = 12x^2 \sin\frac{1}{x} - 5x \cos\frac{1}{x} + \sin\frac{1}{x}, \qquad x \neq 0; \qquad f'(0) = 0;$$

which is discontinuous. Its logarithmic derivative may be derived either by (2) or by (5), and its value when $x = 0$ is zero.

6. Conclusion. The facts and illustrations given above parallel completely the fundamental theorems and illustrations usually given for ordinary derivatives. In deducing them, although the name *logarithmic* derivative has been used, the notion of logarithms has not been employed, and it has been shown that the ideas themselves are independent of the concept of logarithms.

TWO GEOMETRICAL APPLICATIONS OF THE METHOD OF LEAST SQUARES.

By J. L. COOLIDGE, Harvard University.

The two problems discussed in the present paper are taken from a recent work by Vahlen entitled "Konstruktionen und Approximationen."* It seems to me that they are sufficiently interesting and important to merit special attention.

* Leipzig, 1911, pages 125, 126.

Problem 1. *n lines are drawn in a plane which should be concurrent, but are not. If the lines be equally trustworthy, what point should be taken for their expected point of concurrence?*

The solution given (without proof) for this problem is that of Bertot.† It seems rather artificial, and it occurred to me that a more direct solution would have a certain interest. We wish to find such a point that the sum of the squares of its distances from n given lines shall be a minimum. The lines being given in normal form, we have

$$\sum_{i=1}^{i=n} (x \cos \alpha_i + y \sin \alpha_i - p_i)^2 = \text{Minimum.}$$

Equating to zero the partial derivatives as to x and y,

$$\begin{aligned} \sum_{i=1}^{i=n} x \cos^2 \alpha_i + \sum_{i=1}^{i=n} y \cos \alpha_i \sin \alpha_i &= \sum_{i=1}^{i=n} p_i \cos \alpha_i, \\ \sum_{i=1}^{i=n} x \cos \alpha_i \sin \alpha_i + \sum_{i=1}^{i=n} y \sin^2 \alpha_i &= \sum_{i=1}^{i=n} p_i \sin \alpha_i. \end{aligned} \tag{1}$$

Let us suppose that a unit circle has been drawn about the origin as center; the points (1, 0) and (0, 1) shall be called A and B respectively. Circles are drawn on OA and OB as diameters. A perpendicular from the origin on the ith line shall meet that line in P_i, while it meets the OA and OB circles in Q_i and R_i respectively. See Fig. 1. We have for the coördinates of these various points

$$P_i = (p_i \cos \alpha_i,\ p_i \sin \alpha_i), \quad Q_i = (\cos^2 \alpha_i,\ \cos \alpha_i \sin \alpha_i),$$

$$R_i = (\cos \alpha_i \sin \alpha_i,\ \sin^2 \alpha_i).$$

The centre of gravity P of the n points P_i is easily constructed, as are Q, R the centres of gravity of Q_i and R_i. Let their coördinates be

$$P = (\bar{x}, \bar{y}), \quad Q = (x_1, y_1), \quad R = (x_2, y_2), \quad y_1 = x_2.$$

Equations (1) will take the form

$$x_1 x + y_1 y = \bar{x}, \quad x_2 x + y_2 y = \bar{y},$$

$$x = \frac{\begin{vmatrix} \bar{x} & x_2 \\ \bar{y} & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OPR}{\Delta OQR} = \frac{\text{Distance } P - OR}{\text{Distance } Q - OR},$$

$$y = \frac{\begin{vmatrix} x_1 & \bar{x} \\ y_1 & \bar{y} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OQP}{\Delta OQR} = \frac{\text{Distance } P - OQ}{\text{Distance } R - OQ}.$$

† "Solution géométrique du problème de la détermination du lieu le plus probable du navire," *Comptes Rendus*, Vol. LXXXII, 1876, p. 682.

Since we know the unit length already, the values x and y are easily found, and the required point determined.

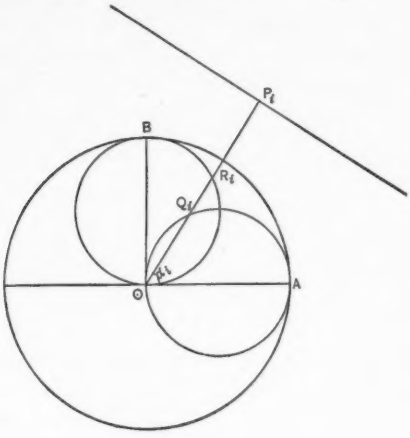


FIG. 1.

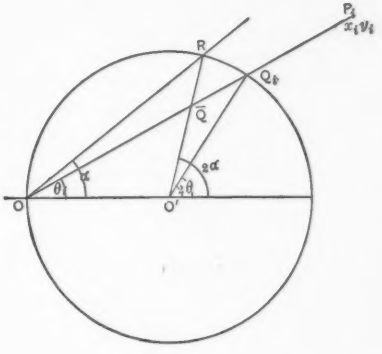


FIG. 2.

Problem 2. *n points are given in the plane which should be collinear, but are not. If the points be equally trustworthy, what line should be taken as their supposed common line?*

Vahlen's remark on this problem is the following:* "Für die konstruktive Ausgleichung einer Geraden aus mehr als zwei Punkten . . . scheinen dem Bertotschen entsprechende einfache Verfahren noch nicht gefunden zu sein. Sie wurden, wenn auch theoretisch von Interesse, doch praktisch schon zu kompliziert sein, um wirklich angewendet werden zu können." Without attempting to settle the obscure question of how complicated a construction may be without losing its practical importance, let us show that this problem is really easier of solution than the other. The given points shall be $(x_1, y_1) \cdots (x_n, y_n)$. We seek a line, the sum of the squares of whose distances from these shall be a minimum. If this line be given in normal form, we have

$$\sum_{i=1}^{i=n} (x_i \cos \alpha + y_i \sin \alpha - p)^2 = \text{Minimum.}$$

Equating to zero the partial derivative as to p, and dividing by n,

$$\frac{\sum_{i=1}^{i=n} x_i}{n} \cos \alpha + \frac{\sum_{i=1}^{i=n} y_i}{n} \sin \alpha - p = 0.$$

This shows that the line must at any rate pass through the center of gravity of all the points. Let us imagine that the origin has been moved to that point.

* Loc. cit., p. 126.

$$\sum_{i=1}^{i=n} (x_i \cos \alpha + y_i \sin \alpha)^2 = \text{Minimum.}$$

Equating to zero the derivative as to α,

$$\sum_{i=1}^{i=n} (y_i^2 - x_i^2) \sin \alpha \cos \alpha = \sum_{i=1}^{i=n} x_i y_i (\sin^2 \alpha - \cos^2 \alpha).$$

Let

$$\frac{y_i}{x_i} = \theta_i,$$

$$\tan 2\alpha = \frac{\sum_{i=1}^{i=n} \sin 2\theta_i}{\sum_{i=1}^{i=n} \cos 2\theta_i}.$$

The origin shall be O, the point (1, 0) shall be O'. We construct a unit circle about O' as center. Let OP_i meet this circle again in Q_i. The coördinates of Q_i in the system *where O' is origin* will be $(\cos 2\theta_i, \sin 2\theta_i)$. Let $\bar{Q}$ be the center of gravity of the points Q_i. The line $O'\bar{Q}$ shall meet the circle in R, that is to say, R is the intersection which lies on the same side of the diameter OO' as does $\bar{Q}$. Then OR is the required line. See Fig. 2.

COMPUTATION FORMULA FOR THE PROBABILITY OF AN EVENT HAPPENING AT LEAST C TIMES IN N TRIALS.

By E. C. MOLINA, New York City.

Many problems in biometry, radioactivity, etc., require for their quantitative solution a convenient formula for computing the probability of an event happening at least c times in n trials; the probability p of its happening in one trial being known. Several relatively simple formulas are given by Poisson in the third chapter of his *Recherches sur la Probabilité des Jugements*, availing himself of a method developed by Laplace in the *Théorie Analytique des Probabilités*. But these formulas explicitly exclude the case where p is very small and $c = np + r$, r not being small compared with np. For this range of values a problem connected with my engineering work forced me early in 1908 to develop the formulas* (5) and (6) given below.

Let P = the required probability, $\alpha = np$, $s = (n - c)/(c + 1)$,

$$F(x) = \frac{2c(1 - x)x^2 + (n - s)sx^4}{[c - (n - s)x]^4}$$

* These formulas have been used for the construction of curves which are in constant use in the Engineering Department of the American Telephone and Telegraph Company. They are published in the hope that others will find them helpful.

and take as a starting point Poisson's exact equation

$$P = \frac{\int_\sigma^\infty Y dy}{\int_0^\infty Y dy}, \qquad \text{where} \qquad Y = \frac{y^{n-c}}{(1+y)^{n+1}} \qquad \text{and} \qquad \sigma = \frac{1-p}{p}.$$

Transforming the numerator by the change of variable $y = (1/x) - 1$ and substituting for the integral in the denominator its well-known value, gives

$$P = \frac{n!}{(c-1)!\,(n-c)!}\int_0^p x^{c-1}(1-x)^{n-c}dx, \tag{1}$$

or, as

$$c = np + r > (n-s)p,$$

we may write

$$P = \frac{n!}{(c-1)!\,(n-c)!}\int_0^p \frac{\partial}{\partial x}\{x^c(1-x)^{n-c-s}\}\frac{(1-x)^{s+1}}{[c-(n-s)x]}dx.$$

By partial integration

$$P = \frac{n!}{(c-1)!\,(n-c)!}\left\{\frac{p^c(1-p)^{n-c+1}}{[c-(n-s)p]} - (n-s)s\int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2}dx\right\}. \tag{2}$$

Also, by partial integration

$$\begin{aligned}\int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2}dx &= \int_0^p \frac{\partial}{\partial x}\{x^c(1-x)^{n-c-s}\}\frac{x^2(1-x)^{s+1}}{[c-(n-s)x]^3}dx \\ &= \frac{p^{c+2}(1-p)^{n-c+1}}{[c-(n-s)p]^3} - \int_0^p x^{c-1}(1-x)^{n-c}F(x)dx.\end{aligned} \tag{3}$$

Now, since $F(x)$ remains finite, does not change sign and is a maximum for $x = p$ as x varies from 0 to p, provided $p < 2/3$,

$$\int_0^p x^{c-1}(1-x)^{n-c}F(x)dx = \lambda F(p)\int_0^p x^{c-1}(1-x)^{n-c}dx, \qquad 0 < \lambda < 1. \tag{4}$$

By (2), (3), (4) and the expressions for s and $F(p)$ we obtain

$$\begin{aligned}P = \frac{n!}{c!\,(n-c)!}p^c(1-p)^{n-c}&\left\{\frac{(1-p)(c+1)}{c+1-(n+1)p}\right\} \\ &\times\left\{1-\frac{(\alpha+p)(\alpha-cp)}{c(c+1-\alpha-p)^2}\right\}\left\{\frac{1}{1-\lambda\varphi}\right\}\end{aligned} \tag{5}$$

where

$$\varphi = \frac{(\alpha+p)(\alpha-cp)\{2(c+1)^2(1-p)+(\alpha+p)(\alpha-cp)\}}{c^2(c+1-\alpha-p)^4} < \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha-p)^4}.$$

For p not greater than 1/30 and P not greater than .01 I found that the values of c and α $(= np)$ are such that φ is negligible. For this range of values, therefore, the factor $1/(1 - \lambda\varphi)$ in the right-hand member of (5) was ignored and a convenient formula for computing P thereby obtained. After plotting several curves it became evident that for a given value of P the corresponding value of c did not change appreciably with changes in the values of p and n, provided the product α $(= np)$ was kept constant. The following formulas were therefore used.

In (5) let n and p approach the limits ∞ and 0 respectively while α remains constant. Then in the limit

$$P = \left\{\frac{1}{1-\lambda\psi}\right\}\left\{\frac{\alpha^c\epsilon^{-\alpha}}{c!}\right\}\left\{\frac{c+1}{c+1-\alpha}\right\}\left\{1 - \frac{\alpha^2}{c(c+1-\alpha)^2}\right\},$$

(6)

$$\psi = \frac{\alpha^2\{2(c+1)^2 + \alpha^2\}}{c^2(c+1-\alpha)^4}.$$

Let P_0, P_1 be the values of P obtained from (6) by taking λ equal to 0 and 1 respectively. Then

(7) $$P_0 < P < P_1.$$

Therefore $(P_1 + P_0)/2$ may be taken as an approximate value of P provided $(P_1 - P_0)/2$ is small compared with P_0. This will be the case if P is not greater than .01 and α/c small. As α/c increases the discrepancy between P and $(P_1 + P_0)/2$ increases, but not indefinitely. This may be proved as follows.

For a given value of P we know (Bernoulli-Laplace Theorem) that α/c approaches the limit 1 as α and c approach ∞. Also, for α and c very large, one of Poisson's formulas is a very close approximation and reduces to

(8) $$P = \frac{2}{\sqrt{\pi}}\int_{\kappa}^{\infty} \epsilon^{-t^2}dt$$

where κ, the lower limit of the integral, is given by the equation

$$\frac{\kappa^2 + (c-1-\alpha)}{(c-1)} = -\log\left\{\frac{\alpha}{c-1}\right\} = -\log\left\{1 - \frac{c-1-\alpha}{c-1}\right\}$$

$$= \frac{(c-1-\alpha)}{(c-1)} + \frac{(c-1-\alpha)^2}{2(c-1)^2} + \frac{(c-1-\alpha)^3}{3(c-1)^3} + \cdots,$$

or, for c very large and $\alpha/c \doteq 1$

(9) $$\kappa^2 = \frac{(c-1-\alpha)^2}{2(c-1)} = \frac{(c+1-\alpha)^2}{2\alpha},$$

but for $\alpha/c \doteq 1$

(10) $$\psi = \frac{\alpha^2\{2(c+1)^2 + \alpha^2\}}{c^2(c+1-\alpha)^4} = \frac{3\alpha^2}{(c+1-\alpha)^4} = \frac{3}{(2\kappa^2)^2}.$$

Therefore, by (10) and any of the well-known tables giving the values of P and κ which satisfy equation (8), we obtain the following figures showing the limits (as $c \doteq \infty$ and $\alpha/c \doteq 1$) of the possible error incurred by taking $P = (P_1 + P_0)/2$.

P	κ^2	ψ	$(P_1 - P_0)/2P_0$
.01	2.72	.101	.056
.001	4.80	.032	.017
.0001	6.92	.016	.008

The short table which follows gives some of the pairs of values of c and α satisfying the three equations $P = .01$, $P = .001$ and $P = .0001$ for the limiting case of $p \doteq 0$, $n \doteq \infty$.

	$P = .0001$		$P = .001$		$P = .01$	
c	α	α/c	α	α/c	α	α/c
1	.0001	.0001	.0010	.001	.010	.010
2	.0142	.0071	.0454	.023	.149	.075
3	.0862	.0289	.191	.064	.436	.145
4	.232	.0580	.429	.107	.823	.206
5	.444	.0888	.739	.148	1.28	.256
6	.714	.119	1.11	.185	1.79	.298
7	1.03	.147	1.52	.217	2.33	.333
8	1.39	.174	1.97	.246	2.91	.364
9	1.78	.198	2.45	.272	3.51	.390
10	2.20	.220	2.96	.296	4.13	.413
12	3.11	.259	4.04	.337	5.43	.452
14	4.11	.293	5.20	.371	6.78	.484
16	5.17	.323	6.41	.401	8.18	.512
18	6.28	.349	7.66	.426	9.62	.534
20	7.44	.372	8.96	.448	11.1	.555
30	13.7	.457	15.9	.530	18.7	.623
40	20.6	.515	23.3	.582	26.8	.670
60	35.4	.590	38.0	.648	43.5	.725
80	50.9	.636	55.2	.690	60.7	.759
100	67.0	.670	71.9	.719	78.2	.782
140	100.2	.716	106.3	.760	114.0	.814
180	134.3	.746	141.4	.786	150.3	.835
200	151.6	.758	159	.795	169	.845
300	240	.800	249	.830	261	.870
400	330	.825	341	.852	355	.887
500	421	.842	434	.868	450	.900
600	513	.855	527	.878	545	.908
700	606	.866	621	.887	640	.914
800	699	.874	715	.894	736	.920
900	793	.881	810	.900	832	.924
1000	887	.887	905	.905	928	.928

BOOK REVIEWS.

W. H. BUSSEY, Chairman of the Committee.

Academic Algebra. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn & Co., Boston, 1913. iv + 442 pages. $1.20.

This book is the second of the Wentworth-Smith series. In it the authors have attempted to provide a high school course which shall cover the topics

named in the various curricula suggested by educational associations. The topics considered valuable but not essential are placed in an appendix. More than this, the authors have sought, by means of the practical problems, to give proper preparation to those who are fitting themselves for a trade. The noteworthy features of the book are the early and simple introduction of graphs with a table of squares and cubes at the end of the book to facilitate computation, the large number of oral problems under each topic and the cumulative reviews at the end of the book. There is also a very brief history of Algebra.

R. R. SHUMWAY.

Higher Algebra. By HERBERT E. HAWKES. Ginn and Company, Boston, 1913. vi + 222 pages. $1.25.

This algebra is an admirable rearrangement and revision of selected chapters from the *Advanced Algebra* published by the same author eight years ago. Pedagogically and typographically the book marks a distinct advance over the earlier work.

As a relatively large number of texts are upon the market dealing with high-school algebra and as more bid fair to appear, it would be highly desirable for the publishers to agree upon a somewhat precise definition of the terms, higher algebra, advanced algebra, college algebra, elementary algebra, and algebra. Our terminology, unfortunately somewhat fixed, seems distinctly inferior to that employed in Europe. A foreign mathematician who chances upon an American "higher algebra" with the content of this one would doubtless be led to inquire into the nature of our "lower algebra." On the other hand, the appearance of our books is decidedly superior, though even this is not an unmixed blessing, for the complete explanations accompanied by numerous diagrams bear mute witness to the fact that many high school teachers require all this paraphernalia to present the subject. Even with the best teachers such complete texts as our American publishers and authors are presenting have their disadvantages, for the pupil is encouraged to believe that the teacher's explanation is of secondary importance. A desirable innovation for test purposes would be a text-book on algebra consisting only of a few formulas, possibly those commonly put in heavy type, together with long sets of exercises. Pupils would be compelled to attend closely to class demonstrations and the teacher would become something more than a commentator.

Eleven chapters are included in this work together with four-place logarithms and tables for the extraction of square and cube roots. The chapter headings are as follows: Introductory review, functions and their graphs, quadratic equations, inequalities, complex numbers, theory of equations, permutations, combinations, and probability, determinants, partial fractions, logarithms, infinite series.

Only minor points would seem to require criticism. The paragraph, page 48, on the "reduced form" $x^2 + px + q = 0$ of the quadratic $ax^2 + bx + c = 0$ seems unnecessary and undesirable. Furthermore, "reduce" is twice used in the introductory discussion of this passage in a different sense from that of the

caption. The omission of any discussion of even the simplest types of simultaneous quadratics is unfortunate in a text explicitly designed for "the student who will continue his mathematics as far as the calculus." In analytic geometry simultaneous quadratics occur as frequently as any topic of algebra. The extension of Horner's method to the computation of larger roots by approximating first to hundreds, then tens, then units, is desirable. No hint of this is given in the text nor is any problem presented which suggests this extension.

Only one historical note is found in the text and that involves a disputed point. Tartaglia arrived at the solution of the cubic before Cardan, as the latter stated in his published work. However Scipio Ferro (died 1526), also cited by Cardan, was prior to both in the solution of the cubic of the form, $x^3 + ax = b$.

L. C. KARPINSKI.

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

388. Proposed by JOSEPH V. COLLINS, Stevens Point, Wis.

On a certain typewriter there is a double scale as follows:

80.....70.....60.....50.....40.....30.....20.....10..... 0
1 ..... 5.....10.....15.....20.....25.....30.....35.....40

It is used to locate headings in the middle of the page. To space a heading one sets the machine at the right stop and with the spacer counts out the number of letters and spaces in the heading. To the reading on the 40 scale where the carriage stops is added the reading of the right stop on the same scale. This number is the one on which to set the carriage pointer on the 80 scale to begin the heading. Show by algebra that the method is correct.

389. Proposed by W. W. BEMAN, Univ. of Michigan.

If $e^{a_1 x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$, prove that

$$na_n = \Sigma_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k}, \qquad \text{or} \qquad n!\,a_n = \Sigma_{k=1}^{k=n} \frac{(n-1)!}{(k-1)!} a_{n-k},$$

which latter form lends itself more readily to computation.

390. Proposed by E. B. ESCOTT, University of Michigan.

Sum the series,

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{5}{16} + \tfrac{8}{32} + \cdots,$$

where each numerator is the sum of the two preceding and the denominators are in geometrical progression.

391. Proposed by C. N. SCHMALL, New York, N. Y.

Show that the roots of the quadratic

$$ax^2 + 2bx + c = 0$$

are imaginary if a, b, c, are in harmonic progression and have the same sign.

GEOMETRY.

417. Proposed by R. P. BAKER, University of Iowa.

Enumerate the points in which the twelve dihedral bisector planes of a tetrahedron meet, find their multiplicity and account for the 220 points which 12 planes in general determine.

418. Proposed by MRS. H. E. TREFETHEN, Waterville, Maine.

The difference of the squares of the two interior diagonals of a cyclic quadrilateral is to twice their rectangle as the distance between their mid-points is to the third diagonal.

419. Proposed by S. LEFSCHETZ, University of Nebraska.

Given a circle and a tangent to it. To find a point on its circumference such that the sum of its distances to the tangent and its point of contact shall be equal to a given length.

(Rouché et de Comberousse: *Géométrie.*)

CALCULUS.

339. Proposed by T. H. GRONWALL, Chicago, Ill.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

340. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu-1)^{1/2}}{1+(\mu-1)^{1/2}},$$

where μ has its usual meaning.

341. Proposed by E. B. ESCOTT, University of Michigan.

Find an expression for the volume of a barrel in terms of the length l, the bung diameter a, and the head diameter b, also an approximate expression when a and b are nearly equal.

MECHANICS.

271. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the center of gravity of the volume formed by the revolution around the x-axis of the area of the curve $y^4 - axy^2 + x^4 = 0$.

272. Proposed by W. J. GREENSTREET, Editor Mathematical Gazette, England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

187. Proposed by L. E. DICKSON, University of Chicago.

Find an amicable number triple by solving one of the equations (other than the last) in the MONTHLY, March, 1913, page 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

188. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find rational values for v, w and x that will satisfy simultaneously the conditions

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2v^2 + m^2n^2(m^2+n^2) = 0, \tag{1}$$

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2+n^2) = 0, \tag{2}$$

$$(m^2+n^2)(v^2+w^2+x^2) - 4m^2n^2x^2 + m^2n^2(m^2+n^2) = 0, \tag{3}$$

m and n being known quantities.

189. Proposed by W. E. HEAL, Washington, D. C.

Develop a formula for the value of x in the equation $p^x + q^x = s^x$, p, q, and s being integral numbers.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

392. Proposed by V. M. SPUNAR, Chicago, Ill.

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R, is equal to the triangle formed by the corresponding points, P', Q', R'.

SOLUTION BY THE PROPOSER.

For the sake of simplicity assume the fixed point O to be the origin and take the tangent and normal to the curve at that point as the co-ordinates of reference. Let the three tangents to the curve be

$$b_1x + a_1y - a_1b_1 = 0, \tag{1}$$

$$b_2x + a_2y - a_2b_2 = 0, \tag{2}$$

$$b_3x + a_3y - a_3b_3 = 0, \tag{3}$$

where (a_i, b_i) are the intercepts of the tangents on the axes. Then, for the double area of the triangle formed by the three tangent lines, we have

$$2\Delta = \frac{[a_1a_2b_3(b_1 - b_2) + a_2a_3b_1(b_2 - b_3) + a_3a_1b_2(b_3 - b_1)]^2}{(a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)}. \tag{4}$$

Again, the double area of the triangle formed by the three points (a_1, b_1), (a_2, b_2), (a_3, b_3) is $2\Delta' = b_1(a_2 - a_3) + b_2(a_3 - a_1) + b_3(a_1 - a_2)$. (5)

To be more general, let

$$\Delta = m\Delta'; \tag{6}$$

or

$$\frac{[a_1a_2b_3(b_1 - b_2) + a_2a_3b_1(b_2 - b_3) + a_3a_1b_2(b_3 - b_1)]^2}{(a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)} = m[b_1(a_2 - a_3) + b_2(a_3 - a_1) + b_3(a_1 - a_2)]. \tag{7}$$

Let the first two tangents be constant and the third one variable with intercepts of the form

$$a_3 = x - y(dx/dy) \quad \text{and} \quad b_3 = y - x(dy/dx)$$

so that

$$b_3 = -a_3p, \quad \text{where} \quad p = dy/dx.$$

Then, substituting a_3 and b_3 from the last expressions in equation (7) after some easy reductions, we get

$$p^4Qx - p^3(C^2x^2 - Lx + Py + N) - p^2(2C^2xy + Mx - Ly - S) - p(C^2y^2 + My - Rx + P) - Ry = 0 = \phi(x, y, p), \tag{8}$$

where the following abbreviations are used:

$$a = a_1a_2, \quad b = b_1b_2, \quad A = a_2 - a_1, \quad B = b_2 - b_1,$$

$$C = a_1b_2 - a_2b_1, \quad D = a_1b_2 + a_2b_1, \quad L = C[ADm + (m+2)aB],$$

$$M = C[BDm + (m+2)bA], \quad N = a(C^2m + aB^2), \quad P = b(C^2m + bA^2),$$

$$Q = maAC, \quad R = mbBC, \quad S = (mC^2D + 2abAB).$$

If we look upon (8) as an algebraical equation in $p = dy/dx$ which has 4 roots p_1, p_2, p_3, p_4, these being functions of x and y, then we are led to the conclusion that through any point in the plane there are *four* directions satisfying the condition proposed. We have, therefore, *four* curves in question.

For the present purpose the equation (8) in p may be considered irreducible. Then, if a singular solution of this equation exists, it must simultaneously satisfy the equations

$$\varphi = 0, \quad \frac{\partial\varphi}{\partial p} = 0, \quad \frac{\partial\varphi}{\partial x} + p\frac{\partial\varphi}{\partial y} = 0.$$

The last equation is *identically zero* in the present case, and we have

$$\frac{d\varphi}{dp} = 4p^3Qx - 3p^2(C^2x^2 - Lx + Py + N) - 2p(2C^2xy + Mx - Ly - S) - (C^2y^2 + My - Rx + P) = 0. \tag{9}$$

Equations (8) and (9), therefore, represent, in parametric form, the singular solution of the differential equation (8), which is, then, the algebraic curve in question.

411. Proposed by C. N. SCHMALL, New York City.

ABCD is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)

II. Solution by the Proposer.

We shall assume that the given rectangle is a square, the problem thus being a special case of problem 382, proposed by R. C. Archibald in the May (1911) number of the Monthly.

Construction: Along *AB* produced lay off *AE* equal to the *given length*. Draw *ED*. Prolong *AD* to *K* so that *DK* = *DE*. On *AK* as a diameter (centre *O*) describe a semi-circle cutting *BC* produced in *Y*. Draw *AY* cutting *DC* in *X*. Then *AXY* is the line required; *i. e.*, the intercept *XY* is of the given length.

Proof: Draw *YH* perpendicular to *AK*. Draw *XK* and *YK*. Then the right triangles *ADX* and *YHK* are equal in all respects (congruent). For the angles *DAX* and *HYK* are equal, and *DA* = *DC* = *HY*.

Hence

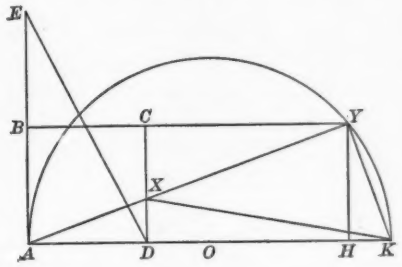
$$AX = YK. \tag{1}$$

Now

$$\overline{XY}^2 + \overline{YK}^2 = \overline{XK}^2 = \overline{XD}^2 + \overline{DK}^2 = \overline{XD}^2 + \overline{DE}^2 \text{ (by constn.)} \tag{2}$$

$$= \overline{XD}^2 + \overline{AD}^2 + \overline{AE}^2 = \overline{AX}^2 + \overline{AE}^2 = \overline{YK}^2 + \overline{AE}^2, \text{ by (1).}$$

Hence, $\overline{XY}^2 = \overline{AE}^2$ and, therefore, $XY = AE =$ the given length.



Also solved by J. Scheffer whose conclusion agrees with that of Professor Shively published in the April number.

H. C. Feemster sent in a solution of 409 too late for notice in April number.

MECHANICS.

250. Proposed by C. N. SCHMALL, New York City.

A smooth circular table in surrounded by a smooth vertical rim. A ball of elasticity e is projected from a point at the rim in a line making an angle θ with the radius through that point. Show that the ball will return to the starting point after the second impact if

$$\tan\theta = \sqrt{\frac{e^3}{e^2 + e + 1}}.$$

SOLUTION BY THE PROPOSER.

Let ABC be the path of the ball and O the center of the table. Let the angles OAB, OBC, be θ, ϕ respectively. Then OBA and OCB are also θ, ϕ, respectively. Let OCA be ψ. Then if the ball returns to A after the second impact the angle CAO will also be ψ. The angles θ, ϕ, ψ, are the angles which the path of the ball makes with the radii passing through it initially and after the first and second impacts. We then have, by elementary principles of impact,

$$\cot\phi = e\cot\theta, \tag{1}$$

$$\cot\psi = e\cot\phi = e^2\cot\theta. \tag{2}$$

But $\theta + \phi + \psi = \pi/2$. Hence $\cot(\theta + \phi + \psi) = 0$.

That is,

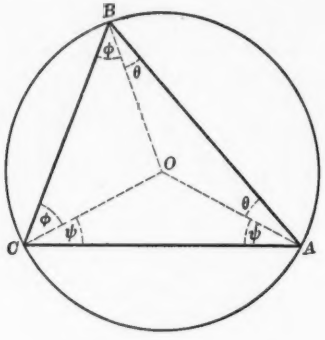
$$\frac{\cot\theta\cot\phi\cot\psi - \cot\theta - \cot\phi - \cot\psi}{\cot\theta\cot\phi + \cot\phi\cot\psi + \cot\psi\cot\theta - 1} = 0.$$

Hence $\cot\theta \cot\phi \cot\psi = \cot\theta + \cot\phi + \cot\psi$, and by (1) and (2),

$$e^3 \cot^3\theta = \cot\theta\,(1 + e + e^2),$$

or,

$$\tan^2\theta = \frac{e^3}{e^2 + e + 1} \qquad \text{and} \qquad \tan\theta = \sqrt{\frac{e^3}{e^2 + e + 1}}.$$



NOTES AND NEWS.

MR. G. W. SMITH, of the University of Colorado, has been appointed assistant in mathematics at the University of Illinois.

DR. B. F. FINKEL, of Drury College, will have charge of the mathematics in the 1913 summer school of the University of Colorado.

The MONTHLY will not be published during the months of July and August. The next issue will appear early in September.

The April number of *The Monist* contains an article by the late HENRY POINCARÉ on "The Relativity of Space."

The first two articles in the latest number of the *Bibliotheca Mathematica* (May 6, 1913) are by PROFESSOR L. C. KARPINSKI. Their titles are as follows: "Hindu numerals among the Arabs," and "The Quadripartitum numerorum of Johannes de Muris."

MR. G. W. HESS, now fellow in mathematics at the University of Michigan, has accepted a call to Shurtleff College, Alton, Illinois, as head of the department of mathematics.

MR. DAVID M. SMITH and MR. DAVID L. STAMY, graduate students at the University of Chicago, have been appointed instructors in mathematics at the Georgia School of Technology, Atlanta, Ga.

The following candidates for the doctorate in mathematics at the University of Chicago have been appointed to University instructorships: MISS MILDRED L.

SANDERSON at the University of Wisconsin, MR. WILLIAM O. BEAL and MR. WILSON L. MISER at the University of Minnesota, and MR. CHARLES R. DINES at Northwestern University.

A book on "Graphical Methods" by CARL RUNGE has just been brought out by the Columbia University Press. It deals with graphical calculation, the graphical representation of functions of one or more independent variables, the graphical methods of the differential and integral calculus.

The prices of full sets of analytical cards, for the German and French editions of the Mathematical Encyklopädie, as issued at this date by the Library of Congress, are as follows: (1) Author set, that is, one copy of each card, German edition $2.60; French edition $1.60; (2) Dictionary catalog set, that is, enough cards for main and secondary entries, German edition $4.08; French edition $1.60.

PROFESSOR C. J. KEYSER, of Columbia University, recently completed a lecture tour arranged by the Sigma Xi Societies of the Universities of Iowa, Minnesota, and Nebraska. The subject of his lecture was "Concerning the figure and the dimensions of the Universe of space."

Marked activity in researches on the history of oriental mathematics is shown by the recent publication of two books which in a way supplement each other, the first by YOSHIO MIKAMI on "The Development of Mathematics in China and Japan" (Leipsig, B. G. Teubner), the second by DAVID EUGENE SMITH and YOSHIO MIKAMI on "The History of Japanese Mathematics" (Chicago, Open Court Publishing Company).

PROFESSOR SAUL EPSTEEN, professor of engineering mathematics at the University of Colorado, has been granted a sabbatic leave of absence. He will spend the year at the state capitol in Denver as commissioner of insurance.

During 1913–14 there will be given at the Ohio State University a "Course for Teachers of Mathematics," devoted to the pedagogy and the history of mathematics. Practice will be given in the presentation of typical subjects and exercises in algebra and geometry and an endeavor will be made to give the student a modern point of view of the field of mathematics. Another course on "Mathematics of Statistics, Finance and Insurance" is offered as an introduction to the mathematical principles of statistical methods, investments and insurance.

The Society of the Sigma Xi which is devoted to the interests of research in the different sciences has begun the publication of a periodical under the title *Sigma Xi Quarterly*. The editorial committee is composed of MCKEEN CATTELL, DAYTON C. MILLER, H. B. WARD, and S. W. WILLISTON. The first number is dated March, 1913, and is largely devoted to the proceedings of the fourteenth convention, held at Cleveland on January 2, 1913.

The 1912–13 edition of *Who's Who in America* includes the names of about 175 members of the American Mathematical Society. It seems therefore safe to conclude that less than one per cent. of the 18,794 sketches of "conspicuous

living Americans" appearing in this work relate to men who have gained distinction by their mathematical work. About one fourth of the names of the present American Mathematical Society appear in *Who's Who*. Viewed from the standpoint of mathematical attainments and proficiency the selection could be much improved.

DR. JAMES W. GLOVER, professor of mathematics and insurance, at the University of Michigan, has been appointed expert special agent of the Bureau of the Census to supervise the preparation of a special volume on vital statistics. Extensive mortality tables are to be prepared, based on the population and vital statistics of the United States. Dr. Glover has also been appointed collaborator to the Office of Public Roads in the Department of Agriculture to assist in the preparation of several bulletins on the various methods of issuing and financing public highway bonds. He will also offer a course of lectures this summer at the University of Oregon summer session on the applications of mathematics to finance, insurance, and statistics.

A reader of the MONTHLY in former years writes that it has become toc abstruse to be read by men of moderate mathematical attainments. An examination of the first five numbers of the present volume will show that 138 pages out of a total of 168 have been devoted to notes, reviews, solution of problems, pedagogical and historical articles and very elementary mathematical papers, while the remaining 30 pages have been given to mathematical articles of a more technical (though elementary) character. It would seem that this is a fair division of subject matter. In this connection see the article in this issue on "Incentives to Mathematical Activity."

"An Introduction to the Mathematical Theory of Heat Conduction with Engineering and Geological Applications" by L. R. INGERSOLL and O. J. ZOBEL, of the University of Wisconsin, is a new book just issued by Ginn & Company. This able text will interest teachers desiring to offer graduate or advanced undergraduate courses on heat conduction. It lays special emphasis upon practical applications but does not enter very extensively into recondite mathematical discussion.

A special commemorative issue of the *Mathematical Gazette*, in honor of the founder, E. M. LANGLEY, and the present editor, W. J. GREENSTREET, was issued in January, 1913, and was edited by A. LODGE. It begins with brief histories of the Association for the Improvement of Geometrical Teaching and of its successor, the Mathematical Association, an organization of teachers and students of elementary mathematics. This association has over 700 members. About three fourths of these live in England, while the remainder are distributed over a number of countries. There are eight in Canada and twenty-one in the United States. SIR GEORGE GREENHILL was elected President for the years 1913 and 1914.

PROFESSOR E. W. HOBSON'S Presidential Address before the Mathematical Association was delivered on January 8, 1913, and was devoted to a consideration

of constructions by means of the compass. Professor Hobson laid stress on two fundamental views of geometry. "On the practical side, geometry is a physical science in which the objects dealt with—points, straight lines, circles, etc.—are physical objects, the relations between which are to be ascertained and described with a view to dealing with actual spatial relations with an accuracy sufficient for all practical purposes. On the theoretical side, geometry is an abstract science." This address appeared in the March, 1913, number of the *Mathematical Gazette*.

The twentieth summer meeting and seventh colloquium of the American Mathematical Society will be held at the University of Wisconsin, Madison, Wis., during the week beginning Monday, September 8, 1913. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and will close on Saturday morning. Courses of lectures will be given as follows:

PROFESSOR L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry." Attention will be limited to the ordinary theory of numbers, and no reference will be made to the general finite field.

PROFESSOR W. F. OSGOOD: "Topics in the theory of analytic functions of several complex variables." The lectures will attempt to give a brief survey of what has been accomplished in the study of some of the more important problems of this branch of analysis.

The Junior Mathematical Club of the University of Chicago has for several years been conducted by the younger graduate students as an adjunct to the Graduate Club and Seminar which is conducted by the faculty and more advanced graduate students. The Junior Club meets fortnightly, and the following papers have been presented during the present year:

"The purpose and work of the club." PROFESSOR H. E. SLAUGHT.

"On the notion of function." PROFESSOR E. H. MOORE.

"Fermat numbers." MR. C. R. DINES.

"On the acoustic figures of a rectangular vibrating membrane." MISS MARTHA MACDONALD.

"On a particular case of an elastic pendulum." MR. YOSHIO ISHIDA.

"An introduction to a function theoretic theory of invariants." PROFESSOR L. E. DICKSON.

"On defining the elliptic functions." MR. W. MISER.

"Some higher plane curves." MISS MARY E. WELLS.

"The radius vector as a function of the time in the hyperbolic case of the problem of two bodies." MR. S. M. SEWALL.

"A problem in Abelian integrals." MR. W. C. VERNON.

"On the projective geometry of the cycloid." MR. J. O. HASSLER.

"On Green's functions." MR. W. C. WESTER.

The University of Edinburgh has resolved upon the establishment of a laboratory for practical instruction in numerical, graphical, and mechanical calculation and analysis, as required in the applied mathematical sciences, and for research in connection with the mathematical department. The courses, under the direction of PROFESSOR E. T. WHITTAKER, embrace the following topics: Differences and interpolation, computations with tables of logarithms, log sines, natural sines, products, quarter-squares, etc., numerical solution of trigonometric problems; Controls for checking accuracy of computations, design of computing-forms; Method of least squares, numerical solutions of systems of linear equations, numerical evaluation of determinants; Curve-fitting, calculation of correlation-coefficients; Analysis of a function into sine and cosine terms (practical Fourier analysis); Analysis for the discovery of periodic constituents in a function (periodogram analysis), practical spherical harmonic analysis, other methods of analysis of functions empirically given; Construction of curves and surfaces, linkages, roulettes, projections, photogrammetry, map-making, graphic solution of numerical equations, graphic and mechanical solution of problems in spherical trigonometry, nomography, applications of triangle of vectors; Use of instruments employed in calculation, especially slide rules, arithmometers, planimeters, integraphs, and harmonic analysers; Numerical evaluation of definite integrals; Numerical solution of differential equations; Numerical evaluation of roots, etc., of transcendental functions; Calculations performed with elliptic functions, arcs on spheroids, etc.; Formation and use of tables of Legendre's and Bessel's functions, the gamma function, error-function, and other transcendental functions; Construction of tables of new functions, and functions not previously tabulated, including automorphic functions, and the parabolic-cylinder and elliptic-cylinder functions.

An important mathematical activity, which should have been mentioned in the leading article of this issue, is the work of the Committee of Fifteen, under the chairmanship of PROFESSOR E. V. HUNTINGTON, of Harvard University, representing the common interests of mathematicians and engineers as voiced in the joint meetings held at Chicago in 1907, under the auspices of the Chicago Section of the American Mathematical Society. The report of this Committee is a syllabus, in book form, of courses in mathematics for students of engineering, published by the Society for the Promotion of Engineering Education. (See the advertisement in this issue.)

PROFESSOR CAJORI'S "History of Logarithms" will be concluded in the September issue. A limited number of copies of the MONTHLY containing this history can be supplied to new subscribers in the order of application. The readers of the MONTHLY will confer a favor upon their friends by calling this information to their attention. Subscriptions should be sent to the Treasurer, B. F. Finkel, Springfield, Missouri.

H. E. SLAUGHT,
Managing Editor.

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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

BY FLORIAN CAJORI, Colorado College.

V. GENERALIZATIONS AND REFINEMENTS EFFECTED DURING THE NINETEENTH CENTURY.

Continued from page 182.

UNIFORMIZATION.

Without the use of the logarithmic function, the interpretation of the power $z^y = w$ becomes complicated even in the simple case when z is negative and y is real. For integral values of y, w then becomes alternately $+$ and $-$; while for rational values m/n, w frequently becomes imaginary, and the pairs of values (z, w), all taken together, yield no continuous curve. At this stage the logarithmic function is introduced to great advantage through the relation $z^{m/n} = e^{m/n \log z}$. Here the n-valued expression $z^{m/n}$ becomes a one-valued function of $\log z$. We say that $z^y = w$ has been *uniformized*. That is w, which in the case when $y = m/n$ is an n-valued function of z, becomes a one-valued function of $\log z$. The function *log z* is a *uniformizing function*. As we look back, we see that the first traces of *uniformization* occurred in the eighteenth century, but not until recently has the process been formally introduced into analysis as a *conscious method*.[1] When in $z^y = w$, y is a real and irrational constant, or when y is a complex constant and z is a complex variable, then w is an infinitely many-valued function of z. In these cases the uniformization $w = z^y = e^{y \log z}$ is still more marked, since w descends from an infinitely many-valued function of z to a one-valued function of $\log z$.

PRINCIPAL VALUES OF POWERS AND LOGARITHMS. PROPOSED NOTATIONS.

Few researches on our topic carried on during the nineteenth century are of greater importance than those introducing the idea of principal values and sug-

[1] See F. Klein, *Elementarmath. v. höheren Standpunkte aus*, T. I, Leipzig, 1908, pp. 296, 352. Consult also E. T. Whittaker, *Modern Analysis*, Cambridge, 1902, p. 338.